Journal of Engineering Mathematics **26**: 153–158, 1992.
H.K. Kuiken and S.W. Rienstra (eds), Problems in Applied, Industrial and Engineering Mathematics.

Causality and the radiation condition

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Abstract. For a hydrostatically stable floating body making small oscillations about a fixed position as a result of external forces and moments, it is shown that the radiation condition implies that the motion at time t depends only upon the forces and moments at times $\leq t$, i.e. that the future does not determine the present.

We wish to consider here one aspect of the motion of a body floating in a heavy fluid. The motions will be assumed to be small enough so that the equations may be linearized. In order to describe the motion of both body and fluid, we shall adopt a right-handed coordinate system with Oz directed against gravity, Ox to the right, and Oy into the paper. The plane Oxy lies in the undisturbed free surface. The small excursions that the body makes about its fixed equilibrium position will be denoted by $\alpha_1, \ldots, \alpha_6$, where $\alpha_1, \alpha_2, \alpha_3$ represent translational displacements and $\alpha_4, \alpha_5, \alpha_6$ angular ones. The dynamical constants of the body will be denoted by m_{ik} where $m_{11} = m_{22} = m_{33} = m$, the mass of the body, and

$$m_{ik} = \int \rho[r^2\delta_{ik} - x_i x_k]\,\mathrm{d}V\,, \quad i, k = 4, 5, 6\,,$$

where ρ is the density distribution of the body, $r^2 = x_i x_i$, and the integral is taken over the body. All other m_{ik} are zero. In the equations to be given below c_{ik} are the hydrostatic coefficients, μ_{ik} are the added masses as defined by Cummins (1962) (i.e., the added masses at infinite frequency), and $L_{ik}(t)$ is a weighting function defined in terms of the velocity potential for the fluid motion. Its definition as well as those of μ_{ik} and c_{ik} may be found in Wehausen (1971 or 1967). An important property of L_{ik} is that it is zero for $t<0$. Let $X_i(t)$ be the force ($i = 1, 2, 3$) or moment ($i = 4, 5, 6$) to which the body is subjected. $X_i(t)$ may be a result of oncoming waves, of wind, or of some other exterior forcing mechanism. We suppose $X_i(t)$ to be absolutely integrable.

In order to accommodate the possibility that only certain modes of motion are allowed, we shall suppose that the subscripts i, j or k may be restricted to some subset A of the integers $1, 2, \ldots, 6$. Then the linearized equations of motion for the body are as follows (see Wehausen, loc. cit.):

$$(m_{ik} + \mu_{ik})\ddot{\alpha}_k(t) + c_{ik}\alpha_k + \int_0^t L_{ik}(t-\tau)\ddot{\alpha}_k(\tau)\,\mathrm{d}\tau = X_i(t)\,, \quad i, k \in A\,. \tag{1}$$

Repeated indices are summed over the integers belonging to A, non-repeated indices take on successively the integers in A.

The equations are a natural candidate for a Fourier or Laplace transform. We shall use the Fourier transform:

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$$X_k(t) = \int_{-\infty}^{\infty} \tilde{X}_k(\sigma)\, e^{-i\sigma t}\, d\sigma . \tag{2}$$

After taking the Fourier transform of the equations of motion, we find the following:

$$\{-\sigma^2[m_{ik} + \mu_{ik}(\sigma)] + c_{ik} - i\sigma\lambda_{ik}(\sigma)\}\tilde{\alpha}_k(\sigma) = \tilde{X}_i(\sigma) , \quad i, k \in A , \tag{3}$$

where

$$\mu_{ik}(\sigma) - \mu_{ik}(\infty) + i\sigma^{-1}\lambda_{ik}(\sigma) = \int_0^{\infty} L_{ik}(\tau)\, e^{i\sigma\tau}\, d\tau . \tag{4}$$

It follows from this that $\mu_{ik}(-\sigma) = \mu_{ik}(\sigma)$ and $\lambda_{ik}(-\sigma) = \lambda_{ik}(\sigma)$. We introduce the following notation:

$$\tilde{M}_{ik} = -\sigma^2[m_{ik} + \mu_{ik}(\sigma)] + c_{ik} , \qquad \tilde{N}_{ik} = \sigma\lambda_{ik}(\sigma) , \qquad \tilde{S}_{ik} = \tilde{M}_{ik} - i\tilde{N}_{ik} . \tag{5}$$

The transformed equation (3) then reads:

$$\tilde{S}_{ik}\tilde{\alpha}_k = \tilde{X}_i , \quad i, k \in A , \tag{6}$$

and its solution is evidently

$$\tilde{\alpha}_i = \tilde{T}_{ik}\tilde{X}_k , \quad i, k \in A , \tag{7}$$

where $\tilde{T} = \tilde{S}^{-1}$, i.e. $\tilde{T}_{ij}\tilde{S}_{jk} = \delta_{ik}$. It now follows easily from the above and from known properties of L_{ik} that $\tilde{S}_{ik}(-\sigma) = \bar{\tilde{S}}_{ik}(\sigma)$, $\tilde{S}_{ki} = \tilde{S}_{ik}$ and similarly for $\tilde{T}_{ik}$. Because of this property of T_{ik} we find

$$\begin{aligned} T_{ik}(t) &= \int_{-\infty}^{\infty} \tilde{T}_{ik}(\sigma)\, e^{-i\sigma t}\, d\sigma \\ &= \int_0^{\infty} [\bar{\tilde{T}}_{ik}(\sigma)\, e^{i\sigma t} + \tilde{T}_{ik}(\sigma)\, e^{-i\sigma t}]\, d\sigma , \end{aligned}$$

i.e. $T_{ik}(t)$ is real. Even though $\tilde{S}_{ik}$ is independent of the choice of A, this is not true for $\tilde{T}_{ik}$ or T_{ik}, which will be different for different choices of A. Although it might be helpful to indicate this, we have not done so in order to avoid a cluttered notation.

Having found $\tilde{\alpha}_i(\sigma)$ above, we may now calculate $\alpha_i(t)$:

$$\begin{aligned} \alpha_i(t) &= \int_{-\infty}^{\infty} \tilde{\alpha}_i(\sigma)\, e^{-i\sigma t}\, d\sigma = \int_{-\infty}^{\infty} \tilde{T}_{ik}\tilde{X}_k\, e^{-i\sigma t}\, d\sigma \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{T}_{ik}(\sigma)\, e^{-i\sigma t}\left[\int_{-\infty}^{\infty} X_k(\tau)\, e^{i\sigma\tau}\, d\tau\right] d\sigma \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\tau\, X_k(\tau) \int_{-\infty}^{\infty} d\sigma\, \tilde{T}_{ik}(\sigma)\, e^{-i\sigma(t-\tau)} \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_k(\tau) T_{ik}(t-\tau)\, d\tau \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} T_{ik}(\tau) X_k(t-\tau)\, d\tau , \quad i, k \in A . \end{aligned} \tag{8}$$

The preceding development is well known and has been reviewed in order to display the last formula for $\alpha_i(t)$, for this formula seems to indicate that α_i at time t depends upon the value of the exciting force X_k at all future as well as all past times unless we can show that $T_{ik}(t)=0$ for all $t<0$. It is this problem that we wish to address and to which we now turn.

First we shall show that the desired property of T_{ik} is equivalent to a certain property of $\tilde{T}_{ik}$. The reasoning is well known and can be found in books on control theory (e.g., Solodovnikov, 1960, pp. 24–28). Consider the transform

$$\tilde{T}_{ik}(\sigma)=\frac{1}{2\pi}\int_{-\infty}^{\infty} T_{ik}(t)\,\mathrm{e}^{\mathrm{i}\sigma t}\,\mathrm{d}t\,.$$

Although heretofore we have thought of σ as being real, we shall now take it to be complex. $\tilde{T}_{ik}(\sigma)$ is then defined in the whole σ-plane. Let us write $\sigma=\rho\,\mathrm{e}^{\mathrm{i}\theta}=\rho(\cos\theta+\mathrm{i}\sin\theta)$. Consider now

$$\int_C \tilde{T}_{ik}(\sigma)\,\mathrm{e}^{-\mathrm{i}\sigma t}\,\mathrm{d}\sigma=\int \tilde{T}_{ik}(R\,\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{e}^{-\mathrm{i}Rt\cos\theta}\,\mathrm{e}^{Rt\sin\theta}Ri\,\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta\,,$$

where the path of integration is either along the semicircle C_+: $\rho=R$, $0<\theta<\pi$, or the semicircle C_-: $\rho=R$, $2\pi>\theta>\pi$. These paths are now completed by paths along the real axis from $-R$ to R. Evidently, as $R\to\infty$ the integral along C_+ converges to zero if $t<0$ and that along C_-converges to zero if $t>0$. It then follows that for $t<0$

$$T_{ik}(t)=\lim_{R\to\infty}\left[\int_{-R}^{R}\tilde{T}_{ik}(\sigma)\,\mathrm{e}^{-\mathrm{i}\sigma t}\,\mathrm{d}\sigma+\int_{C_+}\tilde{T}_{ik}(\sigma)\,\mathrm{e}^{-\mathrm{i}\sigma t}\,\mathrm{d}\sigma\right].$$

If $\tilde{T}_{ik}$ is analytic in the upper half-plane, then $T_{ik}(t)=0$ for $t<0$. The converse of this is also true, i.e. if $T_{ik}(t)=0$ for $t<0$, then $\tilde{T}_{ik}(\sigma)$ is analytic in the upper half-plane.

Our problem has now been transformed to that of showing that $\tilde{T}_{ik}$ is analytic in the upper half-plane. Since $\tilde{T}=\tilde{S}^{-1}$, we shall search for an equivalent property of $\tilde{S}$. Let P_{ik} be the cofactor of the element $\tilde{S}_{ik}$ in the determinant $\det\tilde{S}$ where $i,k\in A$. Then it is known that

$$\tilde{T}_{ik}=P_{ki}/\det\tilde{S}\,,\quad i,k\in A\,. \tag{9}$$

As we shall see, $\tilde{S}$ and hence P_{ik} are analytic in the upper half-plane. Thus what remains to be shown is that $\det\tilde{S}$ has no zeros in the upper half-plane. How do we know that $\tilde{S}$ is analytic in the upper half-plane? From the earlier formulas (4) and (5) defining μ_{ik}, λ_{ik} and $\tilde{S}_{ik}$ it follows that

$$\tilde{S}_{ik}=-\sigma^2\int_0^{\infty}L_{ik}(t)\,\mathrm{e}^{\mathrm{i}\sigma t}\,\mathrm{d}t+c_{ik}-\sigma^2[m_{ik}+\mu_{ik}(\infty)]\,,\quad i,k\in A\,. \tag{10}$$

It has already been mentioned that $L_{ik}(t)=0$ for $t<0$, so that its transform is analytic in the upper half-plane. Since the other terms in the equation above are obviously analytic, $\tilde{S}_{ik}$ is also. We recall that $\tilde{S}$ is a matrix of order between 1 and 6, depending upon A. It will be one of the main-diagonal matrices of the matrix that one would obtain when A consists of all the integers 1 to 6.

We turn now to $\det\tilde{S}$. The matrix $\tilde{S}=\tilde{M}-\mathrm{i}\tilde{N}$ is symmetric but is not hermitian, so that no easy conclusion can be drawn from the fact of symmetry alone. However, the matrix

$$\tilde{S}\tilde{S}^* = [\tilde{M}(\sigma) - i\tilde{N}(\sigma)][\bar{\tilde{M}}^T(\sigma) + i\bar{\tilde{N}}^T(\sigma)] \tag{11}$$

is hermitian, and we shall be able to exploit this fact. Here $\tilde{M}^T$ is the transpose of $\tilde{M}$ and is, of course, equal to $\tilde{M}$ when $\tilde{M}$ is symmetric. We write $\tilde{S}^* = \bar{\tilde{S}}^T$, a usual notation. We note that the product above is hermitian even when $\tilde{S}$ is not symmetric.

Associated with any hermitian matrix $H_{ik} = \bar{H}_{ki}$, $i, k = 1, \ldots, n$, is a so-called hermitian form

$$Q = x_i H_{ik} \bar{x}_k$$

where repeated indices are to be summed from 1 to n. It is easy to show that $Q = \bar{Q}$, so that Q is real. Within this class of forms one distinguishes positive (negative) definite and non-negative (non-positive) definite forms. A non-negative definite form is one such that $Q \geqslant 0$ for any choice of $x_1, \ldots, x_n$; a positive definite form is one such that $Q > 0$ for any choice of $x_1, \ldots, x_n$ except $x_1 = x_2 = \cdots = x_n = 0$. Analogously for the terms in parentheses. A classical theorem about hermitian forms states that such a form is positive definite if and only if all the determinants formed with the first minors along the main diagonal are positive, i.e.,

$$H_{11} > 0\,, \qquad \begin{vmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{vmatrix} > 0, \ldots, \begin{vmatrix} H_{11} & \cdots & H_{1n} \\ H_{21} & \cdots & H_{2n} \\ \vdots & & \vdots \\ H_{n1} & \cdots & H_{nn} \end{vmatrix} > 0\,.$$

It then follows that *all* main-diagonal minors are positive. There is an analogous theorem for negative definite forms and a somewhat more complicated one for non-negative and non-positive definite forms.

Consider now the special hermitian form

$$\begin{aligned} Q &= x_i \tilde{S}_{ij} \bar{\tilde{S}}_{kj} \bar{x}_k = x_i [\tilde{M}_{ij}(\sigma) - i\tilde{N}_{ij}(\sigma)][\bar{\tilde{M}}_{kj}(\sigma) + i\bar{\tilde{N}}_{kj}(\sigma)]\bar{x}_k \\ &= \sum_j |x_i[\tilde{M}_{ij}(\sigma) - i\tilde{N}_{ij}(\sigma)]|^2 \geqslant 0\,, \quad i, j, k \in A\,. \end{aligned} \tag{12}$$

Evidently Q is a sum of squares and hence $Q \geqslant 0$. Suppose that for some particular $\sigma = s + ir$, $r > 0$, there exists a set of complex numbers x_i, $i \in A$, not all zero, such that $Q = \Sigma_j = 0$. Then

$$x_i[\tilde{M}_{ij}(\sigma) - i\tilde{N}_{ij}(\sigma)] = x_i \tilde{S}_{ij} = 0\,, \quad i, j \in A\,,$$

and also

$$\bar{x}_i[\bar{\tilde{M}}_{ij}(\sigma) + i\bar{\tilde{N}}_{ij}(\sigma)] = \bar{x}_i \bar{\tilde{S}}_{ij} = 0\,, \quad i, j \in A\,.$$

The following quadratic forms are then also zero:

$$x_i \tilde{S}_{ij}(\sigma)\bar{x}_j = 0\,, \qquad \bar{x}_i \bar{\tilde{S}}_{ij}(\sigma) x_j = 0\,. \tag{13}$$

Interchanging i and j in the second equation, we obtain

$$x_i \bar{\tilde{S}}_{ji}(\sigma)\bar{x}_j = 0 . \tag{14}$$

After first adding and then subtracting (13) and (14) we obtain:

$$x_i\{\tilde{S}_{ij}(\sigma) + \bar{\tilde{S}}_{ji}(\sigma)\}\bar{x}_j = 0 , \tag{15}$$

$$x_i\{\tilde{S}_{ij}(\sigma) - \bar{\tilde{S}}_{ji}(\sigma)\}\bar{x}_j = 0 , \quad i, j \in A . \tag{16}$$

We now substitute the expression from (10) into (15) and (16):

$$x_i\Big\{-\sigma^2 \int_0^\infty L_{ij}(t)\,\mathrm{e}^{\mathrm{i}\sigma t}\,\mathrm{d}t \mp \sigma^2 \int_0^\infty L_{ji}(t)\,\mathrm{e}^{-\mathrm{i}\sigma t}\,\mathrm{d}t$$
$$+ c_{ij} \pm c_{ji} - \sigma^2[m_{ij} + \mu_{ij}(\infty)] \mp \sigma^2[m_{ji} + \mu_{ji}(\infty)]\Big\}\bar{x}_j = 0 , \tag{17}$$

where the top signs go with (15) and the bottom ones with (16). If we now let $\sigma = s + \mathrm{i}r$, equation (17) becomes the following:

$$x_i\Big\{-(s^2 - r^2 + 2\,\mathrm{i}rs) \int_0^\infty L_{ij}(t)\,\mathrm{e}^{\mathrm{i}st}\,\mathrm{e}^{-rt}\,\mathrm{d}t \mp (s^2 - r^2 - 2\mathrm{i}rs) \int_0^\infty L_{ji}(t)\,\mathrm{e}^{-\mathrm{i}st}\,\mathrm{e}^{-rt}\,\mathrm{d}t$$
$$+ c_{ij} \pm c_{ji} - (s^2 - r^2 + 2\mathrm{i}rs)[m_{ij} + \mu_{ij}(\infty)] \mp (s^2 - r^2 - 2\mathrm{i}rs)[m_{ji} + \mu_{ji}(\infty)]\}\bar{x}_j = 0 . \tag{18}$$

We now exploit the fact that when there is no mean forward motion all matrices are symmetric, i.e., $L_{ij} = L_{ji}$, $c_{ij} = c_{ji}$, etc. It is then easy to deduce the following two equations from (18):

$$x_i\Big\{-2(s^2 - r^2) \int_0^\infty L_{ij}(t)\,\mathrm{e}^{-rt}\cos(st)\,\mathrm{d}t + 4rs \int_0^\infty L_{ij}(t)\,\mathrm{e}^{-rt}\sin(st)\,\mathrm{d}t$$
$$+ 2c_{ij} - 2(s^2 - r^2)[m_{ij} + \mu_{ij}(\infty)]\Big\}\bar{x}_j = 0 , \tag{19}$$

$$x_i\Big\{2(s^2 - r^2) \int_0^\infty L_{ij}(t)\,\mathrm{e}^{-rt}\sin(st)\,\mathrm{d}t + 4rs \int_0^\infty L_{ij}(t)\,\mathrm{e}^{-rt}\cos(st)\,\mathrm{d}t$$
$$+ 4rs[m_{ij} + \mu L_{ij}(\infty)]\Big\}\bar{x}_j = 0 . \tag{20}$$

Between these two equations we may now eliminate first the integrals with $\cos(st)$ and next the integrals with $\sin(st)$ to obtain the following equations:

$$x_i\Big\{(s^2 + r^2)^2 \int_0^\infty L_{ij}(t)\,\mathrm{e}^{-rt}\sin(st)\,\mathrm{d}t + 2rsc_{ij}\Big\}\bar{x}_j = 0 , \tag{21}$$

$$x_i\Big\{(s^2 + r^2)^2 \int_0^\infty L_{ij}(t)\,\mathrm{e}^{-rt}\cos(st)\,\mathrm{d}t + (s^2 - r^2)c_{ij} - (s^2 + r^2)^2[m_{ij} + \mu_{ij}(\infty)]\Big\}\bar{x}_j = 0 . \tag{22}$$

From (4) we easily find

$$L_{ij}(t) = \frac{2}{\pi} \int_0^\infty s'^{-1}\lambda_{ij}(s')\sin(s't)\,\mathrm{d}s'$$

$$= \frac{2}{\pi} \int_0^\infty [\mu_{ij}(s') - \mu_{ij}(\infty)] \cos(s't)\, \mathrm{d}s' . \tag{23}$$

If we substitute the first of these into (21) and integrate with respect to t, we find the following equation:

$$x_i c_{ij} \bar{x}_j + \frac{2}{\pi} \int_0^\infty x_i \lambda_{ij}(s') \bar{x}_j \frac{(r^2 + s^2)}{[r^2 + (s + s')^2][r^2 + (s - s')^2]} \, \mathrm{d}s' = 0 \tag{24}$$

for the hypothesized values of r, s, and x_i, $i \in A$. Equation (22) does not seem to lead to anything useful for our purpose.

We now recall that for a hydrostatically stable floating body the quadratic form $x_i c_{ij} \bar{x}_j$ is non-negative. In fact, if the set A includes only those modes of motion for which there is a hydrostatic restoring force, then it is positive. Moreover, the radiation condition implies that the quadratic form $x_i \lambda_{ij}(s') \bar{x}_j > 0$. Hence the equation (24) cannot hold, that is, the assumption that the hermitian form Q can be zero for some σ in the upper half-plane has led to a contradiction. Thus Q is positive definite and all the main-diagonal determinants of $\tilde{S}\tilde{S}^*$ are >0. But then also det $\tilde{S} \neq 0$, which is what we wanted to prove. A somewhat weaker version of the radiation condition would be sufficient, namely $x_i \lambda_{ij}(s) \bar{x}_j \geqslant 0$ but with the $>$ holding for at least some intervals of s, so that the integral on the right in (24) is >0.

We have shown that the radiation condition implies that $T_{ij}(t) = 0$ if $t < 0$, i.e. that the future does not determine the present, at least in this particular water-wave problem. One would also like to show the converse, that if $T_{ij} = 0$ for $t < 0$ and for i, $j \in A$ for any A, then the radiation condition holds in the weakened form. We have not been able to resolve this problem.

Note that the non-negativeness of the buoyancy quadratic form $x_i c_{ij} \bar{x}_j$ plays an essential role in the proof. On the other hand, the positive definiteness of $x_i m_{ij} \bar{x}_j$ does not seem to be called upon.

Note: The material in this paper was first presented at the First Workshop on Water Waves and Floating Bodies (1986). The author is much indebted to Mr Gyeong Joong Lee of the Department of Naval Architecture of Seoul National University for having pointed out to him not only the inadequacy of the proof presented at the First Workshop but also of a purportedly corrected proof. The present analysis was presented at the Fourth Workshop (1989).

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